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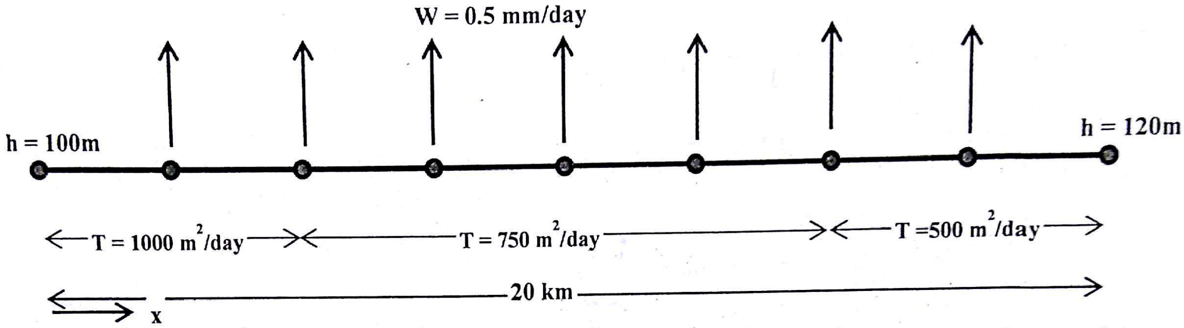
CE 535 / 423 – Groundwater Engineering

TUTORIAL - 5

Q1. A confined aquifer being considered in 1D is given. 20 km length of the aquifer is divided into 9 nodes equally spaced at 2.5 km. Boundary nodes have constant heads, 100 m on the first node and 120 m on the last node. All the interior nodes are subjected to uniform abstraction of 0.5 mm/day. Transmissivity between the first two intervals is 1000 m^2/day, in next four intervals is 750 m^2/day and in the last two intervals, it is 500 m^2/day.

a) Compute steady state nodal heads by solving the appropriate differential equation numerically.



b) Assuming uniform transmissivity, T= 750 m^2/day, on all the nodes, compute the steady state nodal heads by analytical method.
Compare the results obtained in (a) and (b) and sketch the head distribution along x axis.

THOMAS TRI-DIAGONAL MATRIX SOLUTION ALGORITHM

It is a simplified form of Gaussian elimination that can be used to solve tri-diagonal systems of equations. A Tri-diagonal system for n unknowns may be written as

$$a_i x_{i-1} + b_i x_i + c_i x_{i+1} = d_i \qquad i = 0, 2, \ldots\ldots\ldots, n$$

$$\begin{bmatrix} b_0 & c_0 & & & 0 \\ a_1 & b_1 & c_1 & & \\ & a_2 & b_2 & c_2 & \\ & & \ddots & \ddots & c_{n-1} \\ 0 & & & a_n & b_n \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} \qquad \begin{matrix} a_0 = 0 \\ c_n = 0 \end{matrix}$$

$$C_i^* = \begin{cases} \dfrac{c_i}{b_i} & i = 0 \\[2ex] \dfrac{c_i}{b_i - C_{i-1}^* a_i} & i = 1,2,3,\ldots, n-1 \end{cases}$$

$$D_i^* = \begin{cases} \dfrac{d_i}{b_i} & i = 0 \\[2ex] \dfrac{d_i - D_{i-1}^* a_i}{b_i - C_{i-1}^* a_i} & i = 1,2,3,\ldots, n \end{cases}$$

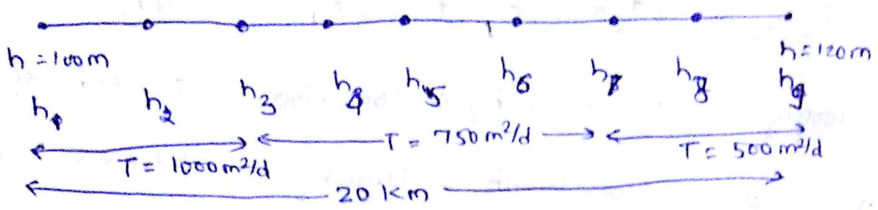
$$x_n = D_n^*$$

$$x_i = D_i^* - C_i^* x_{i+1} \qquad i = n-1, n-2, \ldots\ldots\ldots, 0$$

Name - Avinash Jain

Tutorial - 5 , CE-423 (Ground water Engineering)

(1) Confined aquifer

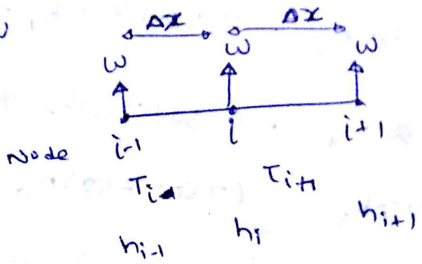


Governing equation : (one-Dimensional)

$$\frac{\partial}{\partial x}\left(T\frac{\partial h}{\partial x}\right) - W = S\frac{\partial h}{\partial t}$$

at Steady State $\frac{\partial h}{\partial t} = 0$

$$\Rightarrow T\frac{\partial^2 h}{\partial x^2} = W$$



$$\left(\frac{\partial h}{\partial x}\right)_i = \frac{h_{i+1} - h_{i-1}}{2\Delta x}$$

$$\left(\frac{\partial^2 h}{\partial x^2}\right)_i = \frac{h_{i-1} - 2h_i + h_{i+1}}{(\Delta x)^2}$$

$$\Rightarrow T_i\left[\frac{h_{i-1} - 2h_i + h_{i+1}}{(\Delta x)^2}\right] = W_i$$ equation at node i

average T at i^{th} node (for Simplicity)

~~For node 1 1000{ h_0 - 2h_1 + h_2~~

$$\frac{\partial}{\partial x}\left(T\frac{\partial h}{\partial x}\right) = \frac{1}{\Delta x}\left[T_{i+1}\left(\frac{h_{i+1} - h_i}{\Delta x}\right) - T_i\left(\frac{h_i - h_{i-1}}{\Delta x}\right)\right]$$

Governing equation

$$\frac{\partial}{\partial x}\left(T\frac{\partial h}{\partial x}\right) = W$$

$$\Rightarrow \frac{1}{\Delta x}\left[T_{i+1}\left(\frac{h_{i+1}}{\Delta x}\right) - \left(\frac{T_i + T_{i+1}}{\Delta x}\right)h_i + \left(\frac{T_i}{\Delta x}\right)h_{i-1}\right]$$

$$\frac{1}{(\Delta x)^2}\left[T_{i+1} h_{i+1} - (T_i + T_{i+1}) h_i + T_i h_{i-1}\right] = \omega(\Delta x)^2$$

for node 2

$$T_3 h_3 - (T_2 + T_3) h_2 + T_2 h_1 = \omega(\Delta x)^2$$

$$\Rightarrow 1000 h_3 - (1000+1000) h_2 + 1000 \times 100 = 0.5 \times 10^{-3} \times (2500)^2$$

$$\Rightarrow 0 - (1000+1000) h_2 + 1000 h_3 = -96875 \rightarrow (1)$$

for node (3) $T_3 h_2 - (T_3 + T_4) h_3 + T_4 h_4 = \omega(\Delta x)^2$

$$\Rightarrow 1000 h_2 - (1000+750) h_3 + 750 h_4 = 3125 \rightarrow (2)$$

for node (4) $750 h_3 - (750+750) h_4 + 750 h_5 = 3125 \rightarrow (3)$

for node (5) $750 h_4 - (750+750) h_5 + 750 h_6 = 3125 \rightarrow (4)$

for node (6) $750 h_5 - (750+750) h_6 + 750 h_7 = 3125 \rightarrow (5)$

for node (7) $750 h_6 - (750+500) h_7 + 500 h_8 = 3125 \rightarrow (6)$

for node (8) $500 h_7 - (500+500) h_8 + 500 h_9 = 3125 \rightarrow (7)$

$$500 h_7 - 1000 h_8 = 3125 - 500 \times 120$$

$$\begin{bmatrix} -2000 & 1000 & 0 & 0 & 0 & 0 & 0 \\ 1000 & -1750 & 750 & 0 & 0 & 0 & 0 \\ 0 & 750 & -1500 & 750 & 0 & 0 & 0 \\ 0 & 0 & 750 & -1500 & 750 & 0 & 0 \\ 0 & 0 & 0 & 750 & -1500 & 750 & 0 \\ 0 & 0 & 0 & 0 & 750 & -1250 & 500 \\ 0 & 0 & 0 & 0 & 0 & 500 & -1000 \end{bmatrix} \begin{bmatrix} h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \end{bmatrix} = \begin{bmatrix} -96875 \\ 3125 \\ 3125 \\ 3125 \\ 3125 \\ 3125 \\ -56875 \end{bmatrix}$$

(3)

$AH = D$

$A = LU$

$$A = \begin{bmatrix} b_1 & c_1 & 0 & 0 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & 0 & 0 & 0 \\ 0 & 0 & a_4 & b_4 & c_4 & 0 & 0 \\ 0 & 0 & 0 & a_5 & b_5 & c_5 & 0 \\ 0 & 0 & 0 & 0 & a_6 & b_6 & c_6 \\ 0 & 0 & 0 & 0 & 0 & a_7 & b_7 \end{bmatrix}$$

$\alpha_1 = -2000$; $\beta_1 = \frac{c_1}{\alpha_1} \Rightarrow \frac{1000}{-2000} \Rightarrow -0.5$

$\alpha_i = b_i - a_i \beta_{i-1}$

$\alpha_i = b_i - a_i \frac{c_{i-1}}{\alpha_{i-1}}$

$\alpha_2 = b_2 - a_2 \frac{c_1}{\alpha_1}$

$= -1750 - (1000)(-0.5)$

$\Rightarrow -1250$; $\beta_2 = \frac{c_2}{\alpha_2} \Rightarrow \frac{750}{-1250} \Rightarrow -0.6$

$\alpha_3 = b_3 - a_3 \beta_2$; $\beta_3 = \frac{c_3}{\alpha_3} \Rightarrow \frac{750}{-1050} \Rightarrow -0.714$

$\Rightarrow -1500 - 750(-0.6)$

$\Rightarrow -1050$

$\alpha_4 = b_4 - a_4 \beta_3$; $\beta_4 = \frac{c_4}{\alpha_4} \Rightarrow -0.778$

$\Rightarrow -1500 - 750(-0.714)$

$= -964.285$

$\alpha_5 = b_5 - a_5 \beta_4$; $\beta_5 = \frac{c_5}{\alpha_5} \Rightarrow -0.818$

$= -1500 - 750(-0.778)$

$\Rightarrow -916.667$

$\alpha_6 = b_6 - a_6 \beta_5$; $\beta_6 = \frac{c_6}{\alpha_6} \Rightarrow -0.785$

$= -1250 - 750(-0.818)$

$= -636.363$

$\alpha_7 = b_7 - a_7 \beta_6$ ~~$\beta_7 = \frac{c_7}{\alpha_7} \Rightarrow 0$~~

$\Rightarrow -1000 - 500(-0.785)$

$\Rightarrow -607.142$

$$AH = D$$

$$\Rightarrow LUH = D \qquad UH = y \; ; \; H = U^{-1}y$$

$$Ly = D \; ; \; y = L^{-1}D$$

$$LU = \underbrace{\begin{bmatrix} \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_2 & \alpha_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_3 & \alpha_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_4 & \alpha_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_5 & \alpha_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_6 & \alpha_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_7 & \alpha_7 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & \beta_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \beta_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \beta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \beta_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \beta_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \beta_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{U}$$

Now, $Ly = D$

$$\begin{bmatrix} -2000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1000 & -1250 & 0 & 0 & 0 & & \\ 0 & 750 & -1050 & 0 & 0 & & \\ 0 & 0 & 750 & -964.28 & 0 & & \\ 0 & 0 & 0 & +750 & & & \\ 0 & 0 & 0 & 750 & & & \\ 0 & 0 & 0 & 0 & & & \end{bmatrix}$$

$$\begin{bmatrix} -2000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1000 & -1250 & 0 & 0 & 0 & 0 & 0 \\ 0 & 750 & -1050 & 0 & 0 & 0 & 0 \\ 0 & 0 & 750 & -964.28 & 0 & 0 & 0 \\ 0 & 0 & 0 & 750 & -916.67 & 0 & 0 \\ 0 & 0 & 0 & 0 & 750 & -636.36 & 0 \\ 0 & 0 & 0 & 0 & 0 & 500 & -602.42 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} = \begin{bmatrix} -96875 \\ 3125 \\ 3125 \\ 3125 \\ 3125 \\ 3125 \\ -56875 \end{bmatrix} \begin{matrix} \rightarrow D_1 \\ \rightarrow D_2 \\ \rightarrow D_3 \\ \rightarrow D_4 \\ \rightarrow D_5 \\ \rightarrow D_6 \\ \rightarrow D_7 \end{matrix}$$

$$y_1 = \frac{D_1}{\alpha_1} = \frac{-96875}{-2000} = 48.4375$$

$$y_2 = \frac{\alpha_1 D_2 - a_2 D_1}{\alpha_1 \alpha_2}$$

$$y_i = \frac{D_i - a_i y_{i-1}}{\alpha_i} \qquad i \in (2, 7)$$

$$y_2 = \frac{3125 - 1000 \times 48.4375}{-1250} = 36.25$$

$$y_3 = \frac{D_3 - a_3 y_2}{\alpha_3} = \frac{3125 - 750 \times 36.25}{-1050} = 22.9167$$

$$y_4 = \frac{D_4 - a_4 y_3}{\alpha_4} = \frac{3125 - 750 \times 22.9167}{-964.2857} = 14.583$$

$$y_5 = \frac{D_5 - a_5 y_4}{\alpha_5} = \frac{3125 - 750 \times 14.853}{-916.67} = 8.5224$$

$$y_6 = \frac{3125 - 750 \times 8.5224}{-636.636} = 5.1313$$

$$y_7 = \frac{-56875 - 500 \times 5.1313}{-607.1424} = 97.902$$

Now, $UH = y$

$$\begin{bmatrix} 1 & -0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -0.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -0.714 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -0.777 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -0.818 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -0.785 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \end{bmatrix} = \begin{bmatrix} 48.4375 \\ 36.25 \\ 22.9167 \\ 14.583 \\ 8.5224 \\ 5.1313 \\ 97.902 \end{bmatrix}$$

So, We have

$$h_8 = 97.902 \text{ m}$$

$$h_7 - 0.785 h_8 = 5.1313$$

$$h_7 = 5.1313 + 0.785 \times 97.902 \Rightarrow 81.9846 \text{m}$$

$$\Rightarrow h_6 - 0.818 h_7 = 8.5224$$

$$h_6 = 0.818 \times 81.9846 + 8.5224 \Rightarrow 75.5854 \text{m}$$

$h_5 = 14.583 + 0.777 \times 75.5854 = 73.3128\,m$

$h_4 = 22.9167 + 0.714 \times 73.3128 = 75.262\,m$

$h_3 = 36.25 + 0.6 \times 75.262 = 81.4072\,m$

$h_2 = 48.4375 + 0.5 \times 81.4072 = 89.1411\,m$

(2) Assume Uniform transmissivity = 750 m²/day on all nodes

For 1D, steady state flow in confined aquifer

$$T \frac{\partial^2 h}{\partial x^2} = W$$

$$\frac{\partial h}{\partial x} = \frac{W x}{T} + C_1$$

$$h = \frac{W x^2}{2T} + C_1 x + C_2$$

$W = 0.5 \times 10^{-3}$ m/d

$W = 5 \times 10^{-4}$ m/d

at $x = 0$, $h = 100\,m$

$\Rightarrow C_2 = 100$

at $x = 20000$, $h = 120\,m$

$$120 = \frac{5 \times 10^{-4}}{2 \times 750} (20000)^2 + C_1 (20000) + 100$$

$\Rightarrow C_1 = -5.667 \times 10^{-3}$

So, $$h = \frac{5 \times 10^{-4}}{2 \times 750} x^2 - 5.667 \times 10^{-3} x + 100$$

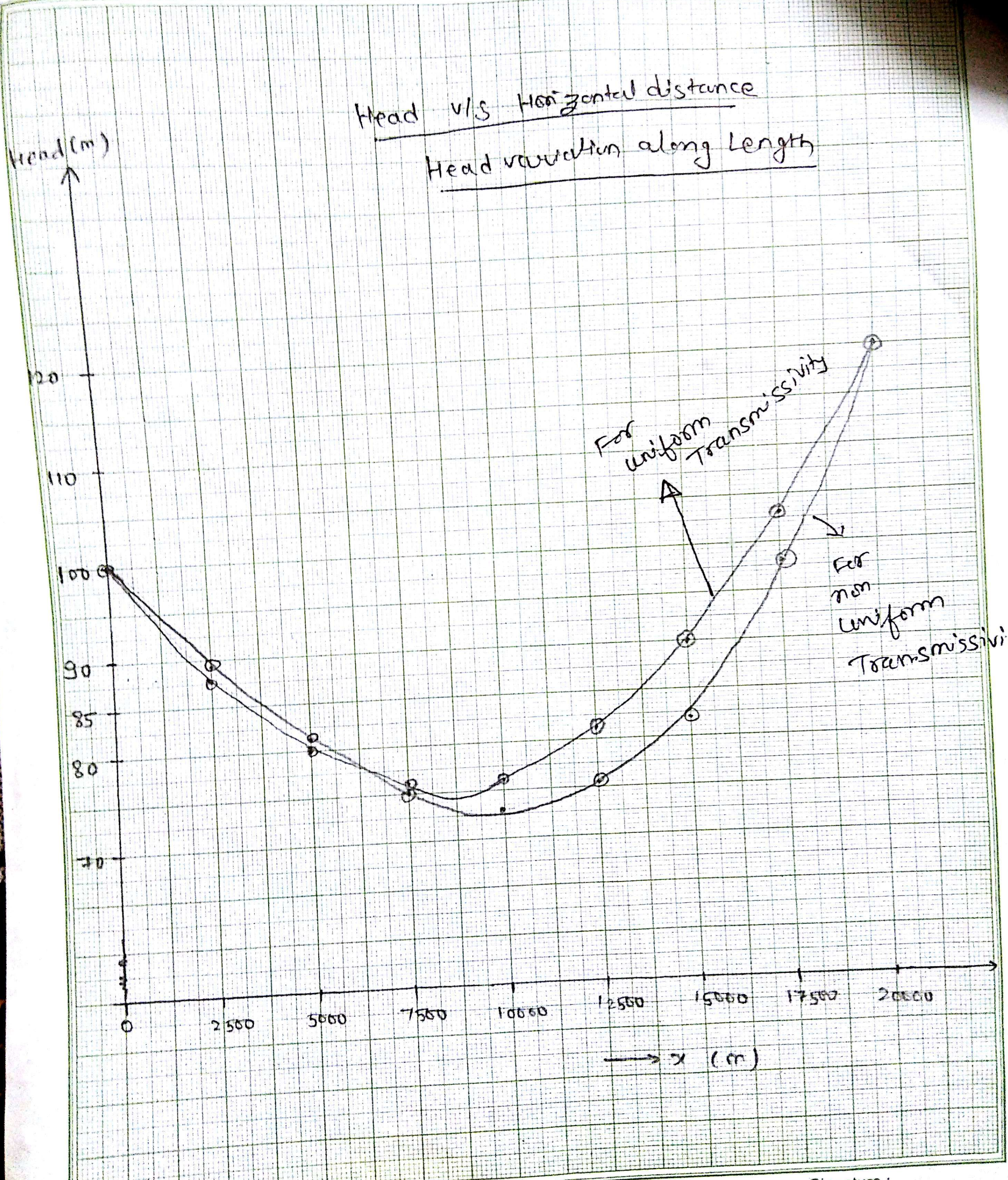
$$\boxed{h = 3.333 \times 10^{-7} x^2 - 5.667 \times 10^{-3} x + 100}$$

So,

Head Profile Along the length (1)

x (m)	h (m) (uniform T)	h (m) (non uniform T)
0	100	100
2500	87.91	89.14
5000	79.99	81.40
7500	76.24	75.26
10000	76.66	73.31
12500	81.24	75.58
15000	89.98	81.98
17500	102.9	97.90
20000	120	120

Name Class Section Roll No.



Measurement Chart of : Remarks : Signature :